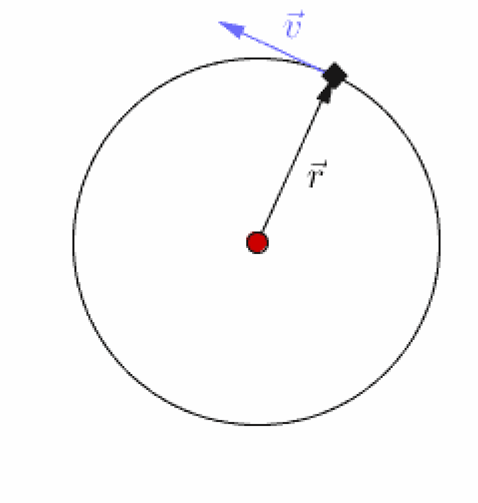
LESSON 3: KINEMATICS II – THE UNIFORM CIRCULAR MOTION

1. Circular motion.



Author: MikeRun, Source: https://commons.wikimedia.org/wiki/File:Uniform-cirular-translation.gif [Requested on December the 6th of 2021]

A uniform circular motion (UCM)), or uniformly accelerated (UACM), can be studied by resorting to the relations deduced in the study of rectilinear motions (KINEMATICS I).

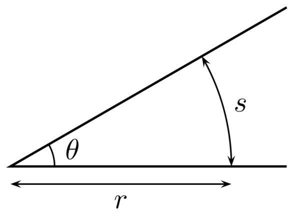
However, the possibility of describing the displacement of the moving point by means of the angle swept by the radius vector pointing to it, opens a new path for its study, exclusive of circular movements, **using magnitudes angular and not linear** magnitudes, that is, using **magnitudes referred to angles and not to the path line**.

2. Angular magnitudes.

2.1. The angles.

We are used to measure the angles in degrees in such a way that an angle within a circle can have a value between 0º and 360º. In fact, angles can have negative values or values greater than 360º. And for more precise measures we can give the values of angles specifying the minutes and the seconds of them, taking into account that 1 degree has 60 minutes (1º = 60’) and 1 minute has 60 seconds (1’ = 60 “). But the right unit of angles in the International System of Units is the radian (rad). *The radian is an angle inscribed in a circle that delimits an arc with a length equal to the radius of the circumference.* The equivalence between sexagesimal degrees and radians is: $\pi\, rad = 180º$.

And the relation between the angle (or angular space) in radians and the corresponding arc of length s in a circumference of radius rs, is the following:



$$\theta = \frac{s}{r}$$

Author: Gustavb, Source: https://commons.wikimedia.org/wiki/File:Angle_measure.svg [Requested on December the 6th of 2021]

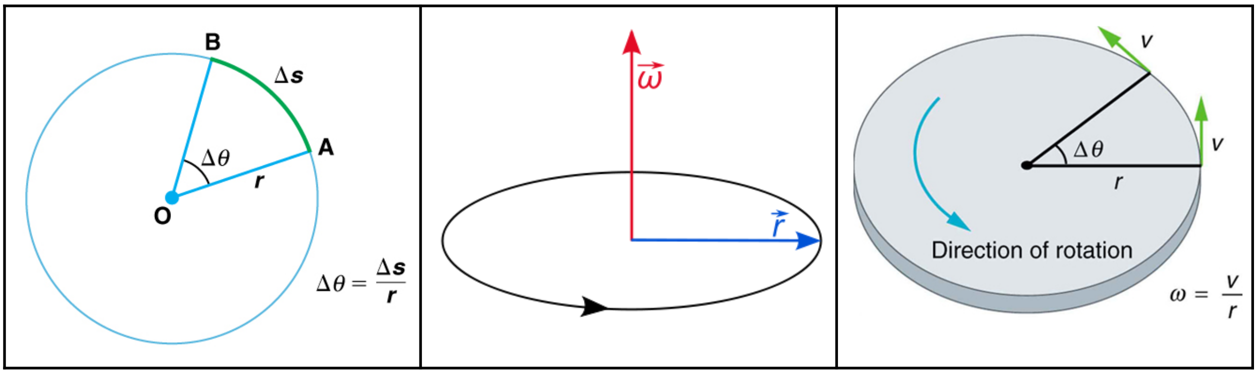
2.2. The angular speed.

In a circular motion, the angular velocity, ω, is defined as the quotient between the angle traveled, θ, measured in radians, and the time taken to sweep it.

Therefore, the units of the angular speed are [ω] = rad/s. Anyway, another typical unit is r.p.m. (revolutions per minute) which are the turns that the mobile makes on a circumference in a minute.

This is the relationship between the angular speed and the angular speed:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\frac{\Delta s}{R}}{\Delta t} = \frac{\Delta s}{R\cdot\Delta t} = \frac{v}{R} \rightarrow v = \omega\cdot R$$



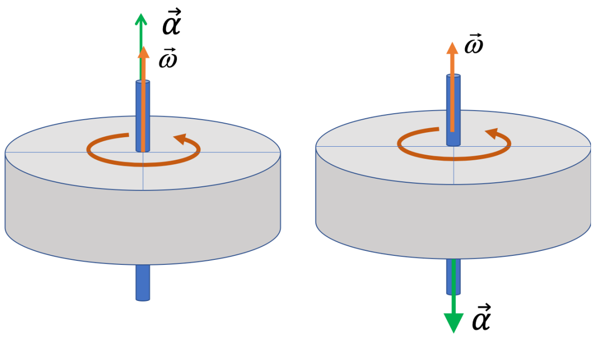
On the left, Author: OpenStax College, College Physics, Source: https://cnx.org/contents/RCFtg_5-@23/Rotation-Angle-and-Angular-Velocity.
In the center, Author: dnet based on raster version released under GFDL, Source: https://commons.wikimedia.org/wiki/File:Angular_velocity.svg.
On the right, Author: OpenStax College, College Physics, Source: shorturl.at/gkqHN [All of them requested on December the 6th of 2021]

2.3. The angular acceleration.

When the angular speed undergoes changes in a circular motion, a new magnitude, the mean angular acceleration, α, is defined as the change in angular velocity per unit time. It is expressed in rad/s^2.

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega-\omega_0}{t}$$

$$(\text{if } t_0 = 0\ s)$$

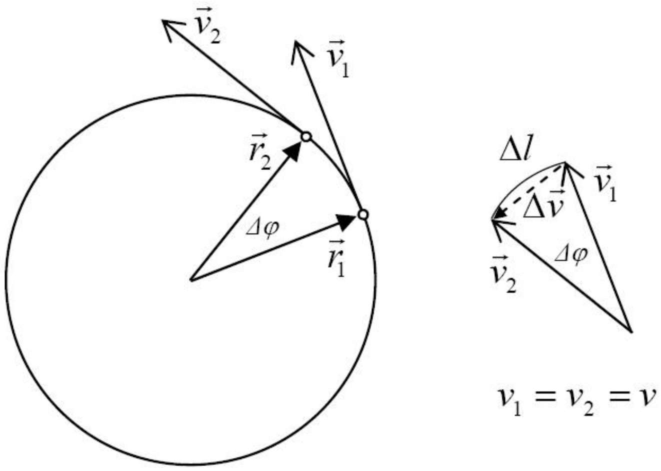


Rotation speeding up Rotation slowing down

Author: Samuel J.Ling, Source: http://www.physicsbootcamp.org/Rotational-Acceleration.html [Requested on December the 7th of 2021]

3. The uniform circular motion (UCM).

The Uniform Circular Motion is one in which the mobile moves in a circular path at a constant speed.



Author: Ilevanat, Source: https://commons.wikimedia.org/wiki/File:Centripetal_acceleration.JPG [Requested on December the 7th of 2021]

In this kind of motion the proper equation associated with it, can be easily derived from the definition of the angular speed. Since,

$$\omega = \frac{\Delta\theta}{\Delta t} \rightarrow \Delta\theta = \omega\cdot\Delta t \rightarrow \theta - \theta_0 = \omega\cdot(t - t_0)$$

Thus, if t_0 = 0 s, it is direct to express the relationship between the swept angle and the time:

$$\theta = \theta_0 + \omega\cdot t$$

On the other hand, it is useful to realize that this can of motion is periodic. This implies some extra magnitudes coming in place for easier description and understanding of it.

- **The period**: It is the time it takes to complete a whole revolution. Because it is a time the units have to be seconds (s), in the International System. It is easy to derive one expression to calculate the period if we replace in the definition of the angular speed the swept angle by the one corresponding to a while revolutions in radians. So:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \rightarrow T = \frac{2\pi}{\omega}$$

- **The frequency**:it is the opposite of the period and it corresponds with the number of revolutions made in one second. Although it can be expressed in s^{-1}, it has its own units in the International System: [f] = Hz (Hertz). So:

$$f = \nu = \frac{1}{T} = \frac{\omega}{2\pi}$$

4. The uniformly accelerated circular motion (UACM).

This movement occurs when a mobile with a circular path increases or decreases its angular velocity constantly, so its angular acceleration remains constant. It is a circular motion of constant angular acceleration α.

Its equations are similar to the UARM equations, in which we change the linear magnitudes by angular magnitudes, in this way we would have:

$$\omega = \omega_0 + \alpha\cdot t \qquad \vartheta = \vartheta_0 + \omega_0\cdot t + \frac{1}{2}\cdot\alpha\cdot t^2 \qquad \omega^2 - \omega_0^2 = 2\cdot\alpha\cdot\Delta\vartheta$$

5. The intrinsic components of the acceleration.

In a curved trajectory the speed can change in two ways: it can change on its module, but it can also change just on its direction. This is the reason why two kinds of accelerations are distinguished, which are called the intrinsic components of the acceleration. We will learn how to imagine, draw and calculate them.

- **The tangent acceleration**: this is the one we already knew from the previous study of motions we did and we applied to rectilinear motions. This acceleration is the one implying a change of the speed on its module. In other words, due to this acceleration the value of the speed increases or decreases. And as its name points out, it is always tangent to the trajectory. The tangent acceleration can be calculated just as:

$$\overline{a}_t = \frac{\Delta v}{\Delta t}\cdot\widehat{u}_t = \frac{v-v_0}{t-t_0}\cdot\widehat{u}_t$$

- **The normal acceleration**: this is the component of the acceleration provoking the movile to bend. So, it is the cause of the mobile to change its direction. This component is oriented along a line perpendicular to the trajectory at the point token into account and points out towards the center of curvature. It can be calculate taking the radius of curvature at a given point of the trajectory as:

$$\overline{a}_n = \frac{v^2}{\rho}\cdot\widehat{u}_n$$

Just to have a rough understanding of how these two components of the acceleration can be derived, we will apply a mathematical rule that, in fact, it is introduced in the first Bachelor in Maths, and determine how to calculate the variation of a magnitude that depends on the product of two other magnitudes. In fact this is what happens with the speed vector: $\overline{v} = v\cdot\widehat{u}_t$.

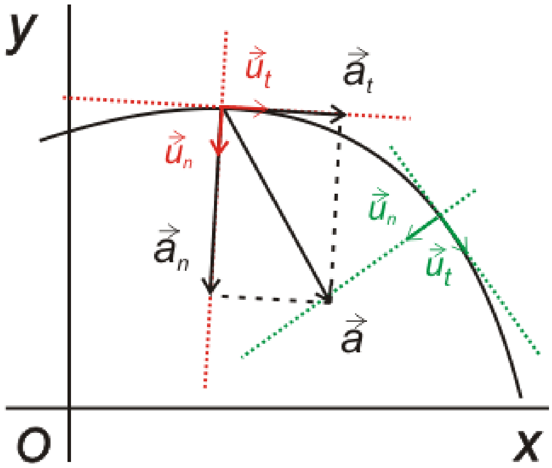
What this mathematical rule, known as the product rule, ennounce is that $\frac{\Delta \bar{v}}{\Delta t} = \frac{\Delta v}{\Delta t} \cdot \widehat{u_t} + v \cdot \frac{\Delta \widehat{u_t}}{\Delta t}$.

It is easy to see that the first terme of the sum is just $\overline{a_t}$.

Let's take a look at the second term now…

And since we don't have yet the right mathematical tools to derive it in a more formal way, we will just imagine that locally, when the mobile changes its directions, it does so as it would be a circular motion with angular speed ω. In other words, the rate of change of the unitary tangent vector with time is just $\frac{\Delta \widehat{u_t}}{\Delta t} = \omega \cdot \widehat{u_n}$.

And as we know from section 2.2 that $v = \omega \cdot R$, and taking the letter ρ for the radius of curvature, it is then direct to obtain $\overline{a_n}$ as it was introduced in the previous page.



Author: Terasa Martín Blas and Ana Serrano Fernández, Source: https://www2.montes.upm.es/dptos/digfa/cfisica/cinematica/cinematica1.htm [Requested on December the12th of 2021]

whereas

- ☐ $\widehat{u_t}$ is the tangent unitary vector.
- ☐ $\widehat{u_n}$ is the normal unitary vector.
- ☐ ρ is the radius of curvature.

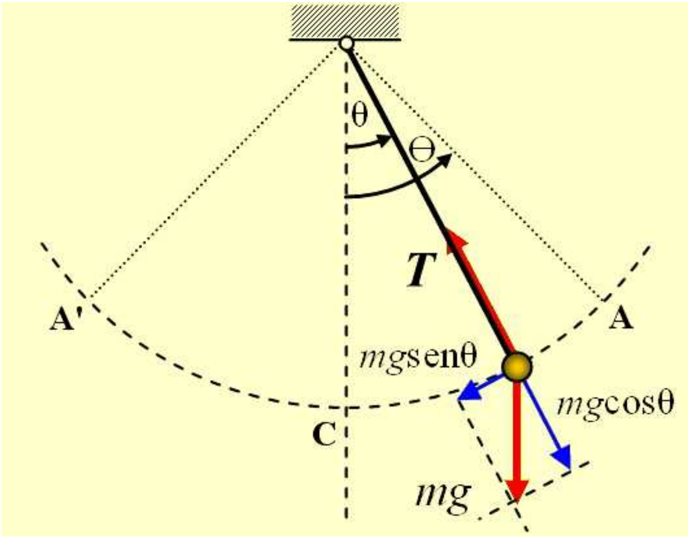
Thus, the total acceleration vector can be calculated as:

$$\bar{a} = \overline{a_t} + \overline{a_n}$$

It is very interesting to highlight that in any uniformly accelerated rectilinear motion (UARM) only the tangent component of the acceleration is different to null while in any uniform circular motion (UCM) only the normal component of the acceleration is different to null.

6. The simple pendulum.

In this section we will derive the mathematical relationship between the length and the period of oscillation of a simple pendulum in a pretty intuitive and simple way.



(Public domain image of) Author: Algarabia, Source: https://commons.wikimedia.org/wiki/File:Pendulosimple.jpg
[Requested on December the12th of 2021]

In order to study the motion of a pendulum in the simplest possible way, it is convenient to choose the axis as to be, at a given point, one tangent to the trajectory, the other aligned along the string and so, perpendicular to the trajectory.

When we draw the forces acting on the pendulum we will find that there are only two: the weight and the tension of the string (both drawn in red).

Since we oriented the axis the way we did, the force that is to be decomposed is the weight. So:

$$P_x = mg \cdot sin\vartheta$$

$$P_y = mg \cdot cos\vartheta$$

Now, applying the condition of all the forces summed along the y axis to be null:

If ϑ is close to 0 rad, $cos\vartheta \simeq 1$ If we consider at the (yellow) point a local constant angular speed.

$$T = P_y = mg \cdot cos\vartheta \simeq mg = m \cdot a_n = m \cdot \frac{v^2}{l} = m \cdot \frac{(\omega \cdot l)^2}{l} = m \cdot \omega^2 \cdot l$$

So, $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l} \rightarrow \frac{2\pi}{T} = \sqrt{\frac{g}{g}} \rightarrow T = 2\pi\sqrt{\frac{l}{g}}$

Thus, $T = 2\pi\sqrt{\frac{l}{g}}$.

And that's way in the study of the pendulum we graphically represented $T^2 = \frac{4\pi^2}{g} \cdot l$ in order to get one straight line with slope $m = \frac{4\pi^2}{g}$ since l corresponds with the x axis and T^2 with y.